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Beta-expansion and continued fraction expansion over formal Laurent series

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Abstract

Let $x \in I$ be an irrational element and $n \geqslant 1$, where I is the unit disc in the field of formal Laurent series $\mathbb{F}((X^{-1}))$, we denote by $k_n(x)$ the number of exact partial quotients in continued fraction expansion of x, given by the first n digits in the β-expansion of x, both expansions are based on $\mathbb{F}((X^{-1}))$. We obtain that

$$\liminf_{n\to+\infty}\frac{k_n(x)}{n}=\frac{\deg\beta}{2Q^*(x)},\qquad \limsup_{n\to+\infty}\frac{k_n(x)}{n}=\frac{\deg\beta}{2Q_*(x)},$$

where $Q^*(x)$, $Q_*(x)$ are the upper and lower constants of x, respectively. Also, a central limit theorem and an iterated logarithm law for $\{k_n(x)\}_{n\geqslant 1}$ are established.

Keywords: β-Expansion; Continued fraction expansion; Laurent series; Haar measure; Finite field

1. Introduction

Let $\mathbb{F}$ be a finite field with q elements. We denote by $\mathbb{F}[X]$ the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(X)$ the field of fractions. Let $\mathbb{F}((X^{-1}))$ be the field of formal Laurent series, i.e.,

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$$\mathbb{F}\big(\big(X^{-1}\big)\big)=\left\{\sum_{n=n_0}^{+\infty}x_nX^{-n}\colon\ x_n\in\mathbb{F}\text{ and }n_0\in\mathbb{Z}\right\}.$$

We call $x\in\mathbb{F}(X)$ be a rational element and $x\in\mathbb{F}((X^{-1}))\setminus F(X)$ be an irrational element. Put $\deg(x)=-\inf\{n\in\mathbb{Z}\colon\ x_n\neq 0\}$ with $x=\sum_{n=n_0}^{+\infty}x_nX^{-n}\in\mathbb{F}((X^{-1}))$, which is called the degree of x and $\deg(0)=-\infty$ with the convention.

Remark 1. $\nu(x)=-\deg(x)$ is an exponential valuation on $\mathbb{F}((X^{-1}))$. We define the norm of x to be $\|x\|=q^{\deg(x)}$, where q is the cardinality of $\mathbb{F}$. With the convention $\|0\|$=0, we have the following:

(1) $\|x\|\geqslant 0$ with $\|x\|=0$ if and only if $x=0$;
(2) $\|xy\|=\|x\|\cdot\|y\|$;
(3) $\|\alpha x+\beta y\|\leqslant\max(\|x\|,\|y\|)$ $(\forall\alpha,\beta\in\mathbb{F})$;
(4) For $\alpha,\beta\in\mathbb{F}$, $\alpha\neq 0$, $\beta\neq 0$, if $\|x\|\neq\|y\|$, then $\|\alpha x+\beta y\|=\max(\|x\|,\|y\|)$.

In other words, $\|\cdot\|$ is a non-Archimedean norm on the field $\mathbb{F}((X^{-1}))$. It is known that $\mathbb{F}((X^{-1}))$ is a complete metric space under the metric ρ defined by $\rho(x,y)=\|x-y\|$.

Let $I=\{x\in\mathbb{F}((X^{-1}))\colon\ \|x\|<1\}$. The set I is isomorphic to $\prod_{n\geqslant 1}\mathbb{F}$ and is an Abel compact group. As a result, there exists a unique normalized Haar measure μ on I given by

$$\mu\big(B\big(a,q^{-r}\big)\big)=q^{-r},$$

where $B(a,q^{-r})=\{x\in\mathbb{F}((X^{-1}))\colon\ \|x-a\|<q^{-r}\}$ is the disc with the center $a\in I$ and radius q^{-r} $(r\in\mathbb{N})$. Note that $\mu(I)=1$ and $(I,\mathscr{B}(I),\mu)$ is a probability space, where $\mathscr{B}(I)$ is Borel field on I.

Every $x\in\mathbb{F}((X^{-1}))$ has a unique (Artin) decomposition (see [1]) as $x=[x]+\{x\}$, where the integral part $[x]$ belongs to $\mathbb{F}[X]$ and the fractional part $\{x\}$ belongs to I.

Remark 2. By the non-Archimedean property of the norm $\|\cdot\|$, we have:

(1) If $x\in B(a,r)$, then $B(a,r)=B(x,r)$, i.e., each point of a disc may be considered as the center of the disc.
(2) If two discs intersect, then the one must contain the other.
(3) For any $0<r<1$, $B(x,r)=B(x,q^{-n_0})$ with $q^{-(n_0+1)}\leqslant r<q^{-n_0}$ and $n_0\in\mathbb{Z}$.

We now cite the β-expansions of formal Laurent series introduced by K. Scheicher [22], M. Hbaib and M. Mkaouar [10] independently.

Let $\beta\in\mathbb{F}((X^{-1}))$ with $\|\beta\|>1$. The β-transformation T_β on I is given as

$$T_\beta x=\beta x-[\beta x].$$

Then every $x\in I$ can be represented by

$$x=\frac{\varepsilon_1(x)}{\beta}+\frac{\varepsilon_2(x)}{\beta^2}+\cdots+\frac{\varepsilon_n(x)}{\beta^n}+\cdots,$$

where $\varepsilon_1(x) = [\beta x]$ and $\varepsilon_n(x) = \varepsilon_1(T_\beta^{n-1}(x))$ for all $n \geqslant 2$, are called the digits of the β-expansion of x. We denote by $(\varepsilon_1(x), \varepsilon_2(x), \ldots)$ the β-expansion of x for simplicity. Since $T_\beta^n x \in I$ for all $x \in I$ and $n \geqslant 1$, we know $\|\varepsilon_n(x)\| < \|\beta\|$ (i.e., $\deg(\varepsilon_n(x)) < \deg \beta$). Conversely, for any given sequence $\{\varepsilon_n\}_{n\geqslant 1}$ with $\varepsilon_n \in \mathbb{F}[X]$ and $\|\varepsilon_n\| < \|\beta\|$ for all $n \geqslant 1$, there exists a unique $x \in I$ such that $\varepsilon_n(x) = \varepsilon_n$ for all $n \geqslant 1$.

Definition 1.1. For any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{F}[X]$ with $\|\varepsilon_i\| < \|\beta\|$ for all $1 \leqslant i \leqslant n$, we call the set

$$J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \bigl\{x \in I\colon\ \varepsilon_i(x) = \varepsilon_i,\ 1 \leqslant i \leqslant n\bigr\}$$

an nth cylinder of the β-expansion.

The following theorem is proved in [16], we present it here for completeness.

Theorem 1.2. *(See [16].) For any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{F}[X]$ with $\|\varepsilon_i\| < \|\beta\|$ $(1 \leqslant i \leqslant n)$,*

$$J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = B\left(\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n}, \frac{1}{\|\beta\|^n}\right).$$

As a consequence, $\mu(J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)) = \|\beta\|^{-n}$.

Proof. Let $x \in B(\frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n}, \frac{1}{\|\beta\|^n})$, then $x = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} + \frac{\theta(x)}{\beta^n}$ with $\|\theta(x)\| < 1$. Since $\theta(x) \in I$, let $\frac{\delta_1}{\beta} + \frac{\delta_2}{\beta^2} + \cdots$ its β-expansion, then $x = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} + \frac{\delta_1}{\beta^{n+1}} + \frac{\delta_2}{\beta^{n+2}} + \cdots$, for the representation $x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^n}$ with $\|\varepsilon_i\| < \|\beta\|$ for all $i \geqslant 1$ is unique, so $\varepsilon_i(x) = \varepsilon_i$ for all $1 \leqslant i \leqslant n$, then $x \in J(\varepsilon_1, \ldots, \varepsilon_n)$. The converse follows immediately by reversing the steps of this proof. Thus we obtain

$$J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = B\left(\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n}, \frac{1}{\|\beta\|^n}\right). \qquad \square$$

In the following we describe the continued fraction expansion of $x \in I$ over the field of the formal Laurent series given by

$$T(x) = \begin{cases} 1/x - [1/x] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Every $x \in I$ has a following unique continued fraction expansion:

$$x = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \cdots}} = \bigl[0; A_1(x), A_2(x), \ldots\bigr],$$

where $A_1(x) = [1/x]$ and $A_n(x) = A_1(T^{n-1}x)$ for $n \geqslant 2$, are called the digits of the continued fraction expansion of x. This expansion was introduced by E. Artin in [1] and has been extensively studied in many papers. For a brief sketch in this framework, see for instance [1] and [3]. In [5] and [20], the authors proved that the Haar measure is invariant for the transformation T. The metrical and ergodic theory of such expansion was studied in [2,9,11,18]. For the connection between such expansion and Diophantine approximation, see for instance [4,9]. Note that every digit is strictly positive degree and the continued fraction expansion of x is finite if and only if $x \in \mathbb{F}(X)$.

Let $\frac{P_n(x)}{Q_n(x)}$ be nth convergent of continued fraction expansion of x, i.e., $\frac{P_n(x)}{Q_n(x)} = [0; A_1(x), \dots, A_n(x)]$.

Proposition 1.3. *(See [9,12,18].) For any $x \in I$ and $n \geqslant 1$,*

(1) $(P_n(x), Q_n(x)) = 1$;
(2) $0 = \deg(Q_0(x)) < \deg(Q_1(x)) < \deg(Q_2(x)) < \cdots$;
(3) $\deg(Q_n(x)) = \sum_{i=1}^n \deg(A_i(x))$.

Theorem 1.4. *(See [18].) For μ-almost all $x \in I$,*

$$\lim_{n\to\infty} \frac{\deg(Q_n(x))}{n} = \frac{q}{q-1}.$$

Denote

$$Q^*(x) = \limsup_{n\to\infty} \frac{\deg(Q_n(x))}{n} \quad \text{and} \quad Q_*(x) = \liminf_{n\to\infty} \frac{\deg(Q_n(x))}{n},$$

which are called the upper and lower constants of x, respectively. Theorem 1.4 implies $Q^*(x) = Q_*(x) = \frac{q}{q-1}$ for μ-almost all $x \in I$.

Definition 1.5. Let $A_1, A_2, \dots, A_n \in \mathbb{F}[X]$ and $n \geqslant 1$, we call the set

$$I(A_1, A_2, \dots, A_n) = \{x \in I\colon\ A_i(x) = A_i,\ 1 \leqslant i \leqslant n\}$$

an nth cylinder of the continued fraction expansion.

Theorem 1.6. *(See [9,12,18].) For any $A_1, A_2, \dots, A_n \in \mathbb{F}[X]$ with strictly positive degree,*

$$I(A_1, A_2, \dots, A_n) = B\left(\frac{P_n}{Q_n},\ q^{-2\sum_{i=1}^n \deg A_i}\right),$$

where $\frac{P_n}{Q_n} = [A_1, A_2, \dots, A_n]$. As a consequence,

$$\mu\big(I(A_1, A_2, \dots, A_n)\big) = q^{-2\sum_{i=1}^n \deg A_i} = q^{-2\deg Q_n}.$$

Let $x \in I$ be an irrational element and denote by $k_n(x)$ the number of exact partial quotients in continued fraction expansion of x, given by the first n digits in the β-expansion of x, i.e.,

$$k_n(x) = \max\{m \geqslant 0\colon\ J\big(\varepsilon_1(x), \dots, \varepsilon_n(x)\big) \subset I\big(A_1(x), \dots, A_m(x)\big)\}.$$

Note that $k_1(x) \leqslant k_2(x) \leqslant \cdots \leqslant k_n(x) \leqslant \cdots$ and $k_n(x) \to \infty$ as $n \to \infty$.

In Section 2 we will give some properties about $k_n(x)$ and the main results, Theorems 2.7, 2.9, 2.10. Section 3 is devoted to establishing Theorem 2.7. The central limit theorem and an iterated logarithm law for $\{k_n(x)\}_{n\geqslant 1}$ will be proved in the last section.

2. Statements of main results

Firstly let us cite some results in the real case. We denote by $k_n^{(\beta)}(x)$ the number of the determined digits between the continued fraction expansion and β-expansion in the real case and by $\mathcal{L}$ the Lebesgue measure on $[0, 1)$. The first result between two expansions is debt to G. Lochs [17], who compared the continued fraction and decimal expansions (i.e., $\beta = 10$) and obtained the following beautiful result.

Theorem 2.1. *(See [17].) For $\mathcal{L}$-almost all $x \in [0, 1)$,*

$$\lim_{n\to\infty} \frac{k_n^{(10)}(x)}{n} = \frac{6\log 2\log 10}{\pi^2} = 0.9702\ldots.$$

For an irrational number $x \in [0, 1)$, let

$$\beta_*(x) = \liminf_{n\to\infty} \frac{\log q_n(x)}{n}, \qquad \beta^*(x) = \limsup_{n\to+\infty} \frac{\log q_n(x)}{n},$$

where $q_n(x)$ is the denominator of the nth convergent of the continued fraction expansion of x. If $\beta_*(x) = \beta^*(x)$, we call the common value $\beta(x)$ the Lévy constant of x.

C. Faivre [8] improved the result of G. Lochs and proved that

Theorem 2.2. *(See [8].) If $x = [0; a_1(x), a_2(x), \ldots]$ has a Lévy constant $\beta(x)$ and partial quotients such that $a_n(x) = O(\alpha^n)$ for all $\alpha > 1$, then*

$$\lim_{n\to\infty} \frac{k_n^{(10)}(x)}{n} = \frac{\log 10}{2\beta(x)}.$$

In fact for $\mathcal{L}$-almost all $x \in [0, 1)$, we have $\beta(x) = \pi^2/(12\log 2)$ by a famous theorem of P. Lévy [14]. Meanwhile, $a_n(x) = O(n^2)$ for $\mathcal{L}$-almost all x from Bernstein's result, see [13, pp. 71, 72]. So the condition posted in Theorem 2.2 is clearly satisfied. So the result of G. Lochs can be got from Theorem 2.2. In [23], the author released the condition on the growth of partial quotients in Theorem 2.2 and got the result concerning every element (i.e., all irrational number); he proved that

Theorem 2.3. *(See [23].) For any irrational $x \in [0, 1)$,*

$$\liminf_{n\to\infty} \frac{k_n^{(10)}(x)}{n} = \frac{\log 10}{2\beta^*(x)}, \qquad \limsup_{n\to\infty} \frac{k_n^{(10)}(x)}{n} = \frac{\log 10}{2\beta_*(x)}.$$

In [15], the authors considered the general case for arbitrary $\beta > 1$ (about the β-expansion of the real case, see [19,21]).

Theorem 2.4. *(See [15].) For $\mathcal{L}$-almost all $x \in [0, 1)$,*

$$\lim_{n\to\infty} \frac{k_n^{(\beta)}(x)}{n} = \frac{6\log 2\log \beta}{\pi^2}.$$

Let $\beta > 1$ be a real number and $\varepsilon(1,\beta) = (\varepsilon_1(1), \varepsilon_2(1), \ldots, \varepsilon_n(1), \ldots)$ be the infinite β-expansion of the number 1 (for details see [15]). Define $l_n = \sup\{k \geqslant 0\colon\ \varepsilon_{n+j}(1) = 0, \text{ for all } 1 \leqslant j \leqslant k\}$. Let

$$A_0 = \Big\{\beta \in (1,+\infty)\colon\ \limsup_{n\to\infty} l_n < +\infty,\ \text{i.e., } \{l_n\} \text{ is bounded}\Big\},$$
$$A_1 = \Big\{\beta \in (1,+\infty)\colon\ \limsup_{n\to\infty} \frac{l_n}{n} = 0\Big\},$$
$$A_2 = \Big\{\beta \in (1,+\infty)\colon\ \limsup_{n\to\infty} \frac{l_n}{n} \neq 0\Big\}.$$

Theorem 2.5. *(See [15].) Let $\beta \in A_0$. Then for any irrational $x \in [0,1)$,*

$$\liminf_{n\to\infty} \frac{k_n^{(\beta)}(x)}{n} = \frac{\log\beta}{2\beta^*(x)}, \qquad \limsup_{n\to\infty} \frac{k_n^{(\beta)}(x)}{n} = \frac{\log\beta}{2\beta_*(x)}.$$

In particular, if β is a Pisot number, the results also hold.

Theorem 2.6. *(See [15].) Let $\beta \in A_1$. Then for all irrational $x \in [0,1)$,*

$$\liminf_{n\to\infty} \frac{k_n^{(\beta)}(x)}{n} = \frac{\log\beta}{2\beta^*(x)}, \tag{2.1}$$

and except a null set E,

$$\limsup_{n\to\infty} \frac{k_n^{(\beta)}(x)}{n} = \frac{\log\beta}{2\beta_*(x)}. \tag{2.2}$$

More precisely, $E = \{x \in [0,1)\colon\ \beta^*(x) = +\infty,\ \beta_*(x) < +\infty\}$.

In [15], the authors conjectured (2.2) can hold for all irrational $x \in [0,1)$ and the results of Theorem 2.6 will be not true for some irrationals if $\beta \in A_2$. In fact, Theorem 2.5 generalizes the result of [23].

Let us turn to the formal Laurent series case. Our results are established for all irrational elements in I with respect to any base $\beta \in \mathbb{F}((X^{-1}))$ with $\|\beta\| > 1$, which is still a conjecture when $\beta \notin A_0$ for the real case. We state our result as follows.

Theorem 2.7. *For any irrational element $x \in I$,*

$$\liminf_{n\to\infty} \frac{k_n(x)}{n} = \frac{\deg\beta}{2Q^*(x)}, \qquad \limsup_{n\to\infty} \frac{k_n(x)}{n} = \frac{\deg\beta}{2Q_*(x)}.$$

As a consequence of Theorems 1.4 and 2.7, we have

Corollary 2.8. *For μ-almost all $x \in I$,*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{(q-1)\deg\beta}{2q}.$$

Finally we get a central limit theorem and an iterated logarithm law for $\{k_n(x)\}_{n\geqslant 1}$, for the corresponding results in the real case, see [7] and [24]. In this paper we always denote the constants

$$E=\frac{(q-1)\deg\beta}{2q} \quad \text{and} \quad \sigma=\frac{\sqrt{(q-1)\deg\beta}}{\sqrt{2}q}.$$

Theorem 2.9. *For any $z\in\mathbb{R}$, we have*

$$\lim_{n\to\infty}\mu\left\{x\in I\colon\ \frac{k_n(x)-nE}{\sigma\sqrt{n}}\leqslant z\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{t^2}{2}}\,dt.$$

Theorem 2.10. *For μ-almost all $x\in I$,*

$$\limsup_{n\to\infty}\frac{k_n(x)-nE}{\sigma\sqrt{2n\log\log n}}=1, \qquad \liminf_{n\to\infty}\frac{k_n(x)-nE}{\sigma\sqrt{2n\log\log n}}=-1.$$

3. Proof of Theorem 2.7

Lemma 3.1. *Let $x\in I$ be an irrational element. Then for any $n\geqslant 1$,*

$$I\big(A_1(x),\ldots,A_{k_n(x)+1}(x)\big)\subset J\big(\varepsilon_1(x),\ldots,\varepsilon_n(x)\big)\subset I\big(A_1(x),\ldots,A_{k_n(x)}(x)\big).$$

Proof. Recall that

$$k_n(x)=\max\big\{m\geqslant 0\colon\ J\big(\varepsilon_1(x),\ldots,\varepsilon_n(x)\big)\subset I\big(a_1(x),\ldots,a_m(x)\big)\big\}.$$

This implies

$$J\big(\varepsilon_1(x),\ldots,\varepsilon_n(x)\big)\subset I\big(A_1(x),\ldots,A_{k_n(x)}(x)\big)$$

and

$$J\big(\varepsilon_1(x),\ldots,\varepsilon_n(x)\big)\not\subset I\big(A_1(x),\ldots,A_{k_n(x)+1}(x)\big).$$

On the other hand, $x\in J(\varepsilon_1(x),\ldots,\varepsilon_n(x))\bigcap I(A_1(x),\ldots,A_{k_n(x)+1}(x))\neq\emptyset$. Then by Remark 2(2), it follows that

$$I\big(A_1(x),\ldots,A_{k_n(x)+1}(x)\big)\subset J\big(\varepsilon_1(x),\ldots,\varepsilon_n(x)\big). \qquad \square$$

Lemma 3.2. *For any irrational element $x\in I$ and $n\geqslant 1$,*

$$\deg\big(Q_{k_n(x)}(x)\big)\leqslant n\frac{\deg\beta}{2}\leqslant\deg\big(Q_{k_n(x)+1}(x)\big).$$

Proof. Lemma 3.1 gives that

$$\mu\big(I\big(A_1(x),\dots,A_{k_n(x)+1}(x)\big)\big) \leqslant \mu\big(J\big(\varepsilon_1(x),\dots,\varepsilon_n(x)\big)\big)$$

and

$$\mu\big(J\big(\varepsilon_1(x),\dots,\varepsilon_n(x)\big)\big) \leqslant \mu\big(I\big(A_1(x),\dots,A_{k_n(x)}(x)\big)\big).$$

By Theorems 1.2 and 1.6, we have

$$q^{-2\deg(Q_{k_n(x)+1}(x))} \leqslant \|\beta\|^{-n} = q^{-n\deg\beta} \leqslant q^{-2\deg(Q_{k_n(x)}(x))}. \qquad \square \tag{3.1}$$

Lemma 3.3. *For any irrational element $x \in I$ and for all $n \geqslant 1$,*

$$k_{n+1}(x) \leqslant k_n(x) + \frac{\deg\beta}{2} + 1.$$

Proof. From Lemma 3.2, we know for all $n \geqslant 1$,

$$\deg\big(Q_{k_n(x)}(x)\big) \leqslant n\frac{\deg\beta}{2} \leqslant \deg\big(Q_{k_n(x)+1}(x)\big).$$

Then it follows that

$$\deg\big(Q_{k_{n+1}(x)}(x)\big) - \deg\big(Q_{k_n(x)+1}(x)\big) \leqslant (n+1)\frac{\deg\beta}{2} - n\frac{\deg\beta}{2} = \frac{\deg\beta}{2}. \tag{3.2}$$

From Proposition 1.3, $\deg(Q_n(x)) = \sum_{i=1}^{n}\deg(A_i(x))$. Since every $A_i(x)$ has a strictly positive degree, from (3.2) we have

$$k_{n+1}(x) - \big(k_n(x)+2\big) + 1 \leqslant \sum_{i=k_n(x)+2}^{k_{n+1}(x)} \deg\big(A_i(x)\big) \leqslant \frac{\deg\beta}{2},$$

that is

$$k_{n+1}(x) \leqslant k_n(x) + \frac{\deg\beta}{2} + 1. \qquad \square$$

Lemma 3.4. *Given any fixed integer $m \geqslant 0$, for any irrational element $x \in I$,*

$$\liminf_{n\to\infty}\frac{\deg(Q_{k_n(x)+m}(x))}{k_n(x)+m} = Q_*(x), \qquad \limsup_{n\to\infty}\frac{\deg(Q_{k_n(x)+m}(x))}{k_n(x)+m} = Q^*(x).$$

Proof. Recall that $k_1(x) \leqslant k_2(x) \leqslant \cdots \leqslant k_n(x) \leqslant \cdots$ and $k_n(x) \to \infty$ as $n \to \infty$. So for any $i \geqslant 1$, there exists $n_i \in \mathbb{N}$ such that

$$k_{n_i}(x) + m \leqslant i \leqslant k_{n_i+1}(x) + m.$$

Then

$$\frac{\deg(Q_{k_{n_i}(x)+m}(x))}{k_{n_i+1}(x)+m} \leqslant \frac{\deg(Q_i(x))}{i} \leqslant \frac{\deg(Q_{k_{n_i+1}(x)+m}(x))}{k_{n_i}+m}. \tag{3.3}$$

By Lemma 3.3 and the left inequality of (3.3),

$$\liminf_{i\to\infty} \frac{\deg(Q_{k_{n_i}(x)+m}(x))}{k_{n_i}(x)+m} \leqslant \liminf_{i\to\infty} \frac{\deg Q_i(x)}{i} = Q_*(x). \tag{3.4}$$

Since $\{k_{n_i}(x)+m\}$ is a subsequence of the sequence $\{k_n(x)+m\}$, from (3.4), we know that

$$\liminf_{n\to\infty} \frac{\deg(Q_{k_n(x)+m}(x))}{k_n(x)+m} \leqslant Q_*(x).$$

By the definition of $Q_*(x)$, the converse inequality is obvious. As a result, we have

$$\liminf_{n\to\infty} \frac{\deg(Q_{k_n(x)+m}(x))}{k_n(x)+m} = Q_*(x).$$

For the other equality on "limsup," in the light of the right part of the inequality (3.3) and Lemma 3.3, the argument is similar. □

Proof of Theorem 2.7. We show the first equality only. For the other it can be done similarly. From Lemma 3.2, we have

$$\limsup_{n\to\infty} \frac{\deg(Q_{k_n(x)}(x))}{k_n(x)} \liminf_{n\to\infty} \frac{k_n(x)}{n} \leqslant \frac{\deg\beta}{2}, \tag{3.5}$$

$$\limsup_{n\to\infty} \frac{\deg(Q_{k_n(x)+1}(x))}{k_n(x)+1} \liminf_{n\to\infty} \frac{k_n(x)+1}{n} \geqslant \frac{\deg\beta}{2}. \tag{3.6}$$

Taking $m=0$, $m=1$ in Lemma 3.4, we have

$$\limsup_{n\to\infty} \frac{\deg(Q_{k_n(x)}(x))}{k_n(x)} = \limsup_{n\to\infty} \frac{\deg(Q_{k_n(x)+1}(x))}{k_n(x)+1} = Q^*(x). \tag{3.7}$$

By (3.5)–(3.7),

$$\liminf_{n\to+\infty} \frac{k_n(x)}{n} = \frac{\deg\beta}{2Q^*(x)}. \qquad \square$$

4. Proofs of Theorems 2.9 and 2.10

Firstly we give the central limit theorem and the law of the iterated logarithm for $\{\deg(Q_n(x))\}_{n\geqslant 1}$, which follows by Proposition 1.3(3) and that the sequence $\{\deg(A_n(\cdot))\}_{n\geqslant 1}$ is independently and identically distributed, also by a growth description for $\{\deg(A_n(x))\}_{n\geqslant 1}$ (for more details, see [18]). In this section, we introduce the constants $E_0 = \frac{q}{q-1}$ and $\sigma_0 = \frac{\sqrt{q}}{q-1}$.

Theorem 4.1. *(See [18].) For any real number z, we have*

$$\lim_{n\to\infty}\mu\left\{x\in I\colon \frac{\deg(Q_n(x))-nE_0}{\sigma_0\sqrt{n}}\leqslant z\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{t^2}{2}}\,dt.$$

Theorem 4.2. *(See [18].) For μ-almost all $x\in I$,*

$$\limsup_{n\to\infty}\frac{\deg(Q_n(x))-nE_0}{\sigma_0\sqrt{2n\log\log n}}=1,\qquad \liminf_{n\to\infty}\frac{\deg(Q_n(x))-nE_0}{\sigma_0\sqrt{2n\log\log n}}=-1.$$

Theorem 4.3. *(See [18].) For μ-almost all $x\in I$,*

$$\limsup_{n\to\infty}\frac{\deg(A_n(x))}{\log n}=\frac{1}{\log q}.$$

In the following we will give the proof of Theorem 2.9. Firstly we state some lemmas.

Lemma 4.4. *For any real number z, we have*

$$\lim_{n\to\infty}\mu\left\{x\in I\colon \frac{\deg(Q_{k_n(x)}(x))-k_n(x)E_0}{\sigma_0\sqrt{k_n(x)}}\leqslant z\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{t^2}{2}}\,dt.$$

Proof. By Theorem 4.1 we can get the result since $\{k_n(x)\}_{n\geqslant 1}$ is a subsequence of $\{n\}_{n\geqslant 1}$. □

Lemma 4.5. *For μ-almost all $x\in I$,*

$$\lim_{n\to\infty}\frac{\deg(Q_{k_n(x)}(x))-n\frac{\deg\beta}{2}}{\sqrt{n}}=0.$$

Proof. Let us denote

$$W_n(x)=\frac{\deg(Q_{k_n(x)}(x))-n\frac{\deg\beta}{2}}{\sqrt{n}},\qquad W_n'(x)=\frac{\deg(Q_{k_n(x)+1}(x))-n\frac{\deg\beta}{2}}{\sqrt{n}}.$$

By Lemma 3.2, for all irrational element $x\in I$ and $n\geqslant 1$,

$$\deg\bigl(Q_{k_n(x)}(x)\bigr)\leqslant n\frac{\deg\beta}{2}\leqslant\deg\bigl(Q_{k_n(x)+1}(x)\bigr).$$

This implies $\limsup_{n\to\infty}W_n(x)\leqslant 0$ and $\liminf_{n\to\infty}W_n'(x)\geqslant 0$.

From Theorem 4.3, for μ-almost all $x\in I$, $\limsup_{n\to\infty}\frac{\deg(A_n(x))}{\sqrt{n}}=0$. Since $\frac{\deg(A_n(x))}{\sqrt{n}}>0$ for all $n\geqslant 1$, then

$$\limsup_{n\to\infty}\frac{\deg(A_{k_n(x)+1}(x))}{\sqrt{k_n(x)+1}}=0 \tag{4.1}$$

since $\{k_n(x)+1\}$ is a subsequence of $\{n\}$. Then for μ-almost all $x \in I$,

$$\limsup_{n\to\infty} \frac{\deg(A_{k_n(x)+1}(x))}{\sqrt{n}} = 0 \tag{4.2}$$

by Corollary 2.8 and (4.1).

Since $\deg(Q_{k_n(x)}(x)) = \deg(Q_{k_n(x)+1}(x)) - \deg(A_{k_n(x)+1}(x))$, then

$$W_n(x) = W_n'(x) - \frac{\deg(A_{k_n(x)+1}(x))}{\sqrt{n}}.$$

It follows that

$$\liminf_{n\to\infty} W_n(x) \geqslant \liminf_{n\to\infty} W_n'(x) - \limsup \frac{\deg(A_{k_n(x)+1}(x))}{\sqrt{n}} \geqslant 0$$

for $\liminf_{n\to\infty} W_n'(x) \geqslant 0$ and (4.2). Combining with $\limsup_{n\to\infty} W_n(x) \leqslant 0$, we have for μ-almost all $x \in I$,

$$\lim_{n\to\infty} W_n(x) = 0. \qquad \square$$

Proof of Theorem 2.9. Let $x \in I$ be an irrational element. Put

$$X_n(x) = -\frac{\deg(Q_{k_n(x)}(x)) - k_n(x)E_0}{\sigma_0\sqrt{k_n(x)}},$$

$$Y_n(x) = \frac{\sigma_0}{\sigma E_0}\sqrt{\frac{k_n(x)}{n}} \quad \text{and} \quad Z_n(x) = \frac{\deg(Q_{k_n(x)}(x)) - n\frac{\deg\beta}{2}}{\sigma E_0\sqrt{n}}.$$

Then

$$\frac{k_n(x) - nE}{\sigma\sqrt{n}} = X_n(x)Y_n(x) + Z_n(x).$$

By Lemma 4.4, $\{X_n\}$ converges to the standard normal distribution in distribution. By Corollary 2.8, $\{Y_n\}$ μ-almost surely converges to the number 1. By Lemma 4.5, $\{Z_n\}$ μ-almost surely converges to the number 0. Therefore, $\{X_nY_n + Z_n\}$ converges to the standard normal distribution in distribution (for more probability theory see [6]). That is, for any real number z,

$$\lim_{n\to\infty} \mu\left\{x \in I\colon \frac{k_n(x) - nE}{\sigma\sqrt{n}} \leqslant z\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt. \qquad \square$$

Next we will give the proof of Theorem 2.10.

Lemma 4.6. *For any fixed integer $m \geqslant 0$, we have for μ-almost all $x \in I$,*

$$\limsup_{n\to\infty} \frac{\deg(Q_{k_n(x)+m}(x)) - (k_n(x)+m)E_0}{\sigma_0\sqrt{2(k_n(x)+m)\log\log(k_n(x)+m)}} = 1 \tag{4.3}$$

and

$$\liminf_{n\to\infty} \frac{\deg(Q_{k_n(x)+m}(x)) - (k_n(x)+m)E_0}{\sigma_0\sqrt{2(k_n(x)+m)\log\log(k_n(x)+m)}} = -1. \tag{4.4}$$

Proof. By Lemma 3.3 and Theorem 4.2, using similar method in the proof of Lemma 3.4, we can get the result. □

Proof of Theorem 2.10. Let $x \in I$ be an irrational element. Put

$$X_n(x) = -\frac{\deg(Q_{k_n(x)}(x)) - k_n(x)E_0}{\sigma_0\sqrt{2k_n(x)\log\log k_n(x)}},$$

$$Y_n(x) = \frac{\sigma_0}{\sigma E_0}\sqrt{\frac{2k_n(x)\log\log k_n(x)}{2n\log\log n}} \quad \text{and} \quad Z_n(x) = \frac{\deg(Q_{k_n(x)}(x)) - n\frac{\deg\beta}{2}}{\sigma E_0\sqrt{2n\log\log n}}.$$

Then

$$\frac{k_n(x) - nE}{\sigma\sqrt{2n\log\log n}} = X_n(x)Y_n(x) + Z_n(x). \tag{4.5}$$

Let B_1, B_2 and B_3 be the exceptional sets that Corollary 2.8, Lemma 4.5 and Lemma 4.6 with $m = 0$ do not hold, respectively. Let $A = I\backslash(B_1 \cup B_2 \cup B_3)$, then $\mu(A) = 0$.

For any $x \in A$, $\lim_{n\to\infty} Y_n(x) = 1$ by Corollary 2.8, $\lim_{n\to\infty} Z_n(x) = 0$ by Lemma 4.5. Taking $m = 0$ in Lemma 4.6, we have

$$\limsup_{n\to\infty} X_n(x) = 1 \quad \text{and} \quad \liminf_{n\to\infty} X_n(x) = -1.$$

Combining with (4.5) we get

$$\limsup_{n\to\infty} \frac{k_n(x) - nE}{\sigma\sqrt{2n\log\log n}} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{k_n(x) - nE}{\sigma\sqrt{2n\log\log n}} = -1. \qquad \square$$

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